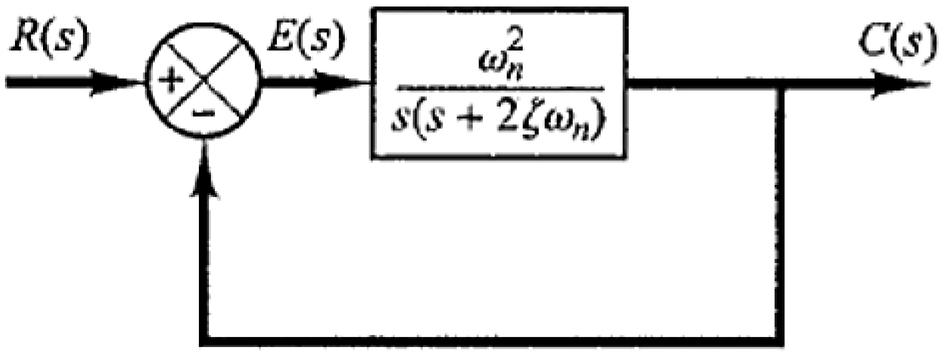
Lecture 6: Time Domain Analysis (Part 2)

6.1 Second-Order Systems:

Consider the following block diagram of closed-loop control system. Here, an open-loop transfer function, ω_n^2 / s(s+2$\xi\omega_n$) is connected with a unity negative feedback.



The transfer function of the closed loop control system with a unity negative feedback is:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substituting $G(s) = \frac{\omega_n{}^2}{s(s+2\zeta\omega_n)}$ in the above equation, we get the standard form of the second-order system (the standard closed-loop transfer function of the second-order system):

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n{}^2}{s(s+2\zeta\omega_n)}\right)}{1+\left(\frac{\omega_n{}^2}{s(s+2\zeta\omega_n)}\right)} = \frac{\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$$

6.1.1 Basic Form:

$$\frac{C(s)}{R(s)} = \frac{\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$$

6.1.2 Key Parameters:

$\boldsymbol{\omega_n}$**:** undamped natural frequency, and ζ: (Zeta) damping ratio or damping factor.

6.2 The Dynamic Behavior of The Second-Order System:

The dynamic behavior of the second-order system can be described in terms of two parameters:

1) Undamped natural frequency $\boldsymbol{\omega_n}$**.**
2) Damping ratio or damping factor ζ**.**

The characteristic equation is:

$$s^2 + 2\zeta\omega_n s + {\omega_n}^2 = 0$$

By comparing the characteristic equation with the quadratic equation $a\,x^2 + b\,x + c = 0$ and using the quadratic formula:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1, \qquad b = 2\zeta\omega_n, \qquad and \quad c = {\omega_n}^2$

Thus, the roots of characteristic equation are:

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4{\omega_n}^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{4{\omega_n}^2(\zeta^2 - 1)}}{2}$$

$$\frac{-2\zeta\omega_n \pm 2\omega_n\sqrt{\zeta^2 - 1}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

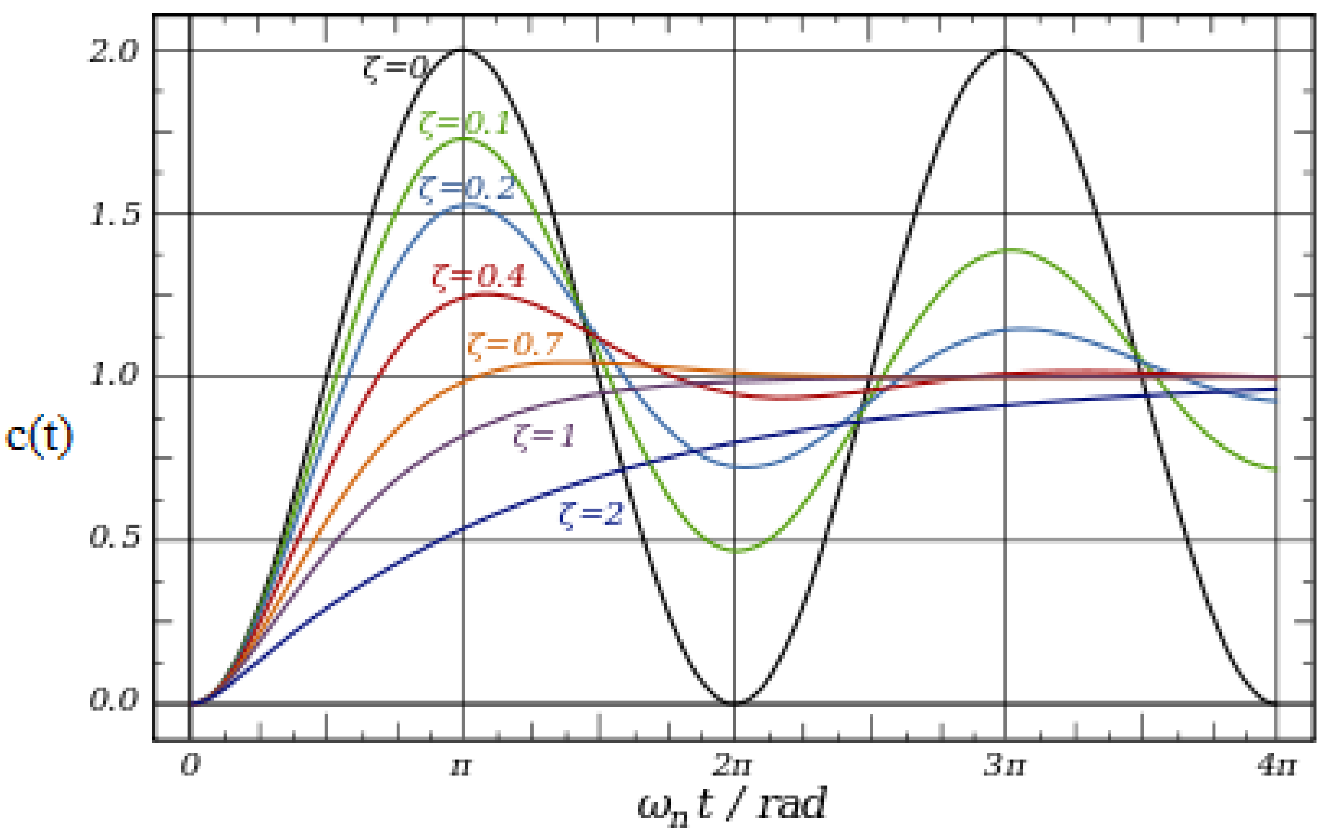
$$\therefore s_1, s_2 = -\zeta\omega_n \pm \omega_d$$

Where $\boldsymbol{\omega_d = \omega_n\sqrt{\zeta^2 - 1}}$: the damped natural frequency.

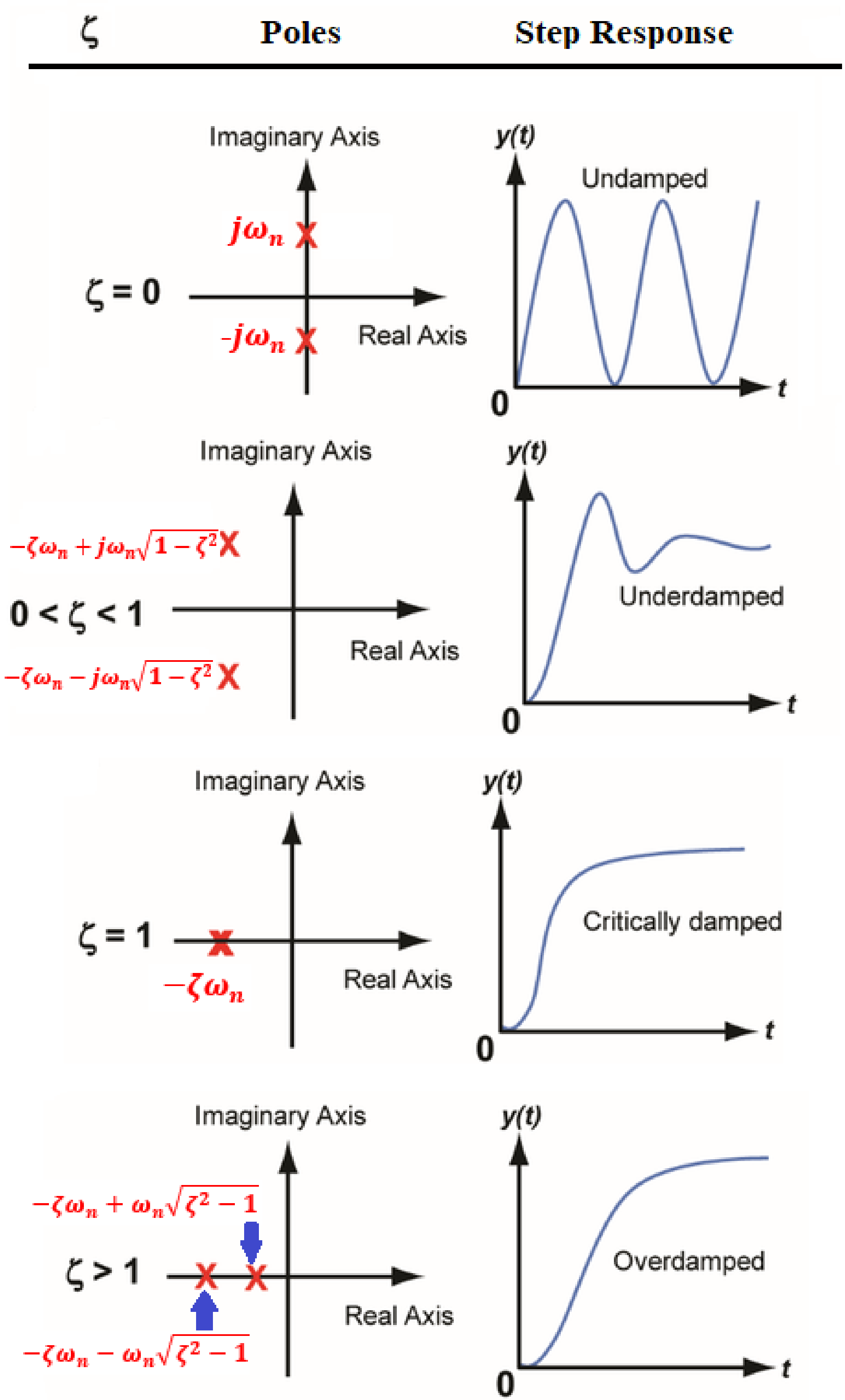
The dynamic behavior of the second-order system changes fundamentally with ζ, as follows:

- If $0 < \zeta < 1$ then the closed-loop poles are complex conjugates and lie in the left-half s-plane. Here, the system is called under damped and the transient response is exponentially damped sinusoidal function.
- If $\zeta = 0$ then the closed-loop poles are imaginary. Here, the system is called undamped and the transient does not die out.
- If $\zeta = 1$ then the closed-loop poles are real and equal. Here, the system is called critically damped.
- If $\zeta > 1$ then the closed-loop poles are distinct real poles (real but not equal). Here, the system is called over damped.

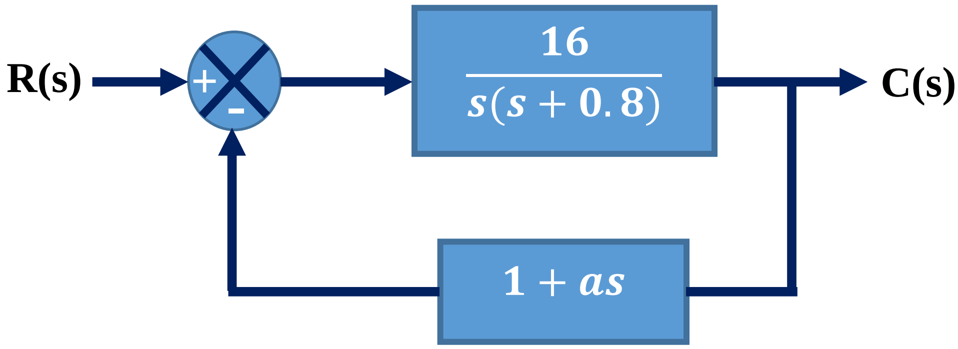
The standard response curves for second-order systems are shown in Figure below.



The various cases of second-order response are shown in Figure below.



Example 1: **Consider the system shown in figure below, determine the value of 'a' such that the damping ratio is 0.5.**



Solution: The transfer function of the closed-loop system is given by:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Substitute **G(s)** and $\boldsymbol{H(s)}$ in the above equation yields:

$$\frac{C(s)}{R(s)} = \frac{\frac{16}{s(s+0.8)}}{1 + [\frac{16}{s(s+0.8)}\ (1+as)]}$$

Multiply the numerator and denominator by $(s(s+0.8))$ yields:

$$\frac{C(s)}{R(s)} = \frac{16}{s(s+0.8) + 16(1+as)}$$

$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + 0.8s + 16 + 16\,as} = \frac{16}{s^2 + (0.8 + 16\,a)s + 16}$$

Compare above equation with the standard second-order equation:

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

$${\omega_n}^2 = 16 \implies \omega_n = 4\,rad/sec$$

$$2\zeta\omega_n = 0.8 + 16\,a \implies 2 \times 0.5 \times 4 = 0.8 + 16\,a$$

$$4 = 0.8 + 16\,a \implies a = \frac{4 - 0.8}{16}$$

$$a = 0.2$$

Example 2: **For the second-order differential equation given below, determine the values of 'a' and 'k' such that the damping ratio is 0.5 and the natural angular frequency is 4 rad/sec.**

$$\boldsymbol{k\,x(t) = \frac{d^2y(t)}{dt^2} + (2 + ak)\frac{d\,y(t)}{dt} + k\,y(t)}$$

Solution:

Take Laplace transform for both sides of the second-order differential equation, yields:

$$k\,X(s) = s^2\,Y(s) + (2 + ak)s\,Y(s) + k\;\;Y(s)$$

$$k\,X(s) = (s^2\; + (2 + ak)s\; + k)\;\;Y(s)$$

$$\therefore \frac{Y(s)}{X(s)} = \frac{k}{s^2\; + (2 + ak)s\; + k}$$

Compare above equation with the standard second-order equation:

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

$${\omega_n}^2 = k = 4^2 = 16$$

$$2\zeta\omega_n = 2 + ak \;\Longrightarrow 2 \times 0.5 \times 4 = 2 + 16\,a$$

$$4 = 2 + 16\,a \;\Longrightarrow a = \frac{4 - 2}{16}$$

$$a = 0.125$$